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Stability of a fluid layer with uniform heat generation and convection boundary conditions

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INTRODUCTION

The success of finding the exact critical Rayleigh number for a fluid layer with unstable linear temperature profile attracted many researchers to the stability theory [1]. Sparrow and his coworkers have investigated the instability when the layer has uniform heat source [2]. Examples of heat generation in fluid layer are abundant, for instance, electric current in a semiconducting fluid such as glass and electrolyte generates Joulean heating. Heating of flowing water in a solar collector and radiative cooling of molten glass in a forehearth are such examples too. While Sparrow *et al.* investigated the case of fixed boundary temperatures and rigid surfaces only, practical cases are mostly with convection boundary condition (with rigid or free upper surface), which means the upper surface has certain thermal resistance with the environment and the lower surface also has an insulation layer. Thus, the general case of convection boundary condition and hydraulic upper surface condition needs to be investigated extensively, however, it has not been studied for the past 30 years. This was the motivation for this study. Though conventional techniques have been employed in this study, the results are believed to be of interest to the industries manufacturing glass, batteries, solar collectors, etc.

The study of convection initiation in a fluid layer of unstable temperature profile provides many interesting phenomenological aspects such as the effects of enclosure geometry, surface tension, electric field, sudden heating, property variation, radiative heating, etc. Many published literature is available. The linear stability theory is most frequently employed in the analysis. It is also used here to find the neutral stability limit, i.e. the limit of decaying fluctuation without oscillation. The principle of exchange of stability is not proved, however, for layers with internal heating so that the results given in this paper are not a complete description of the stability.

PROBLEM DEFINITION, FORMULATION AND NUMERICAL SCHEME

Figure 1 shows a stationary fluid layer with uniform internal heat source. The upper surface may be hydraulically rigid or free and the lower surface is rigid. The upper surface is exchanging heat with the neighboring gas with a given heat transfer coefficient. The bottom also has a fixed heat transfer coefficient and surrounding temperature (these are not necessarily the same as the top surface values). The stationary state is maintained by a certain energy balance between the heat generation and the heat transfers with the upper and the lower surrounding temperatures.

The governing equations to describe the motion of the fluid layer are equations of continuity, momentum and energy balance. With Boussinesq approximation, they are written as,

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{1}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = \nu \frac{\partial^2 u_i}{\partial x_j x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + g\beta(T - T_0)\delta_i^3 \tag{2}$$

$$\frac{\partial T}{\partial t} + \frac{\partial (u_j T)}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j x_j} + \frac{\dot{q}}{\rho c_p} \tag{3}$$

where the repeated indices denote summation over $j = 1$ to 3 and the Kronecker delta δ_i^3 is unity when $i = 3$ and zero otherwise. The boundary conditions are; for velocity components u_1 and u_2 which are parallel to the surface, $u_i = 0$ (rigid) or $\partial u_i/\partial z = 0$ (free). For u_3 (or w), $u_3 = 0$ at the surface whether it is free or rigid. For T, $k\,\partial T/\partial z = -h(T - T_{\infty U})$ at $z = L$ and $k\,\partial T/\partial z = h(T - T_{\infty B})$ at $z = 0$. When the layer is stable, no motion is observed and the temperature distribution is determined solely by conduction. This state is denoted by subscript s (for stationary). When the stationary state is unstable, any perturbed fluctuation (with superscript *) grows and gives an infinitesimal change on the temperature field. The velocity $u_i = u_{i_s} + u_i^*$ and the temperature $T = T_s + T^*$ are introduced into the above equations and arranged to give the governing equations for the perturbation quantities. A lengthy manipulation of the governing equations following the procedure of Pellew and Southwell [1] leads to the following 6th-order partial differential equation for the z-direction velocity perturbation w^*

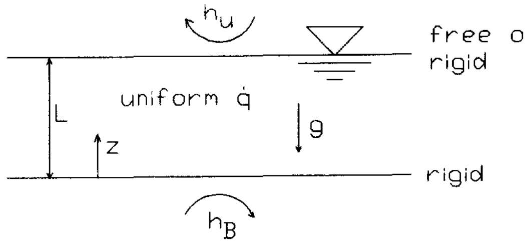

Fig. 1. Schematic diagram of the problem.

NOMENCLATURE

a	wavenumber
$b_n^{(i)}$	coefficient of the power series for $f^{(i)}(Z)$
Bi	Biot number, hL/k
c_p	specific heat
C_i	coefficient of $f^{(i)}(Z)$
$f^{(i)}(Z)$	homogeneous solutions of F-equation, equation (7), $i = 0, 1, \dots 5$
F	amplitude function of the normal mode for w^*
g	gravitational constant
G	exponential factor
h	heat transfer coefficient
H	amplitude function of the normal mode for T^*
k	thermal conductivity
L	depth of the fluid layer
Ns	dimensionless heat source strength $\dot{q}L^2/2k(T_B - T_U)$
p	pressure
$\dot{q}$	heat source strength per unit volume
Ra	modified Rayleigh number $g\beta(T_{max} - T_U)(L - z_{max})^3/\alpha\nu$
Ra^0	original Rayleigh number $g\beta(T_B - T_U)L^3/\alpha\nu$
t	time
T	temperature
T_s^0	dimensionless steady temperature, $T_s/(T_U - T_B)$
u	velocity
w	z-direction velocity
x, y	horizontal coordinates
z	vertical coordinate
Z	$= z/L$.

Greek symbols

α	thermal diffusivity
β	volumetric thermal expansion coefficient
δ_i^j	Kronecker delta ($= 1$ when $i = j$ and 0 otherwise)
ν	kinematic viscosity
ρ	fluid density
σ	the exponential growth rate.

Subscripts

B, U	bottom and top surfaces
c	critical
i, j	components along Cartesian coordinates
max	maximum temperature
s	stationary state
o	reference
∞	surrounding.

Superscript

*	perturbed quantity.

$$\left(\frac{\partial}{\partial t} - \alpha\nabla^2\right)\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w^* + g\beta\frac{\partial T_s}{\partial_z}\nabla_{xy}^2 w^* = 0 \quad (4)$$

where the operator Δ_{xy}^2 means Laplacian for x and y, i.e. $(\partial^2/\partial x^2) + (\partial^2/\partial y^2)$.

The fluctuation velocity and temperature are assumed to have product form $w^* = (\alpha/L)F(z)G(x,y)\exp(\sigma t)$ and $T^* = (T_B - T_U)H(z)G(x,y)\exp(\sigma t)$ where the exponential factor G and amplitudes F and H are all dimensionless quantities. The factor G satisfies,

$$\nabla_{xy}^2 G + \left(\frac{a}{L}\right)^2 G = 0. \quad (5)$$

Functions satisfying equation (5) can express any two-dimensional periodic pattern of convection roll when viewed from above. The wavenumber a appears as G represents a convection pattern infinitely repeated on the x–y plane. A typical example of normal modes satisfied by G is the real part of $e^{i(a_x x + a_y y)/L}$ where $a^2 = a_x^2 + a_y^2$. When $a_x = 0$, G expresses roll cells stretching its axis along the x-axis with periodic repetition in the y-direction. In finding the stability, the real part of σ is assigned zero and the critical Rayleigh number for non-oscillatory mode is obtained by further assigning the imaginary part of σ zero. It can also be shown that H and F have the following relation.

$$\frac{g\beta(T_B - T_U)L^3}{\nu\alpha}H = \frac{1}{a^2}\left\{\frac{d^4F}{dZ^4} - 2a^2\frac{d^2F}{dZ^2} + a^4F\right\} \quad (6)$$

where the vertical coordinate has been normalized as $Z = z/L$ and we have the original Rayleigh number Ra^0 defined as $g\beta(T_B - T_U)L^3/\alpha\nu$. The expression for w^* is introduced into (4) together with equation (5) to give a 6th-order ordinary differential equation for F in dimensionless form,

$$\frac{d^6F}{dZ^6} - 3a^2\frac{d^4F}{dZ^4} + 3a^4\frac{\partial^2 F}{\partial Z^2} + \left(-a^6 + a^2 Ra^0\frac{\partial T_s^0}{\partial Z}\right)F = 0 \quad (7)$$

where the dimensionless temperature T_s^0 is defined as $T_s^0/(T_U - T_B)$. It is readily obtained from equation (3) and the gradient is found to be

$$\frac{\partial T_s^0}{\partial Z} = [1 - Ns(1 - 2Z)] \quad (8)$$

where the dimensionless heat source Ns is defined as, following Sparrow *et al.* [2],

$$Ns = \frac{\dot{q}L^2}{2k(T_B - T_U)}. \quad (9)$$

The boundary conditions for F are given as follows. At the bottom, the velocity is zero.

$$F(0) = 0. \quad (10)$$

Also, the x- and y-direction velocities are zero and thus from the continuity, $\partial w^*/\partial z = 0$, i.e.

$$dF(0)/dZ = 0. \quad (11)$$

Since $w^* = 0$ at the top,

$$F(1) = 0. \quad (12)$$

If the upper surface is rigid,

$$dF(1)/dZ = 0. \quad (13)$$

However, if it is free, then since the vertical direction gradients of the horizontal velocity components are zero at the top, we get, by differentiating the continuity equation with Z,

$$d^2F(1)/dZ^2 = 0. \tag{14}$$

The upper and lower surfaces are subject to convection boundary conditions, i.e.

$$\frac{\partial H(1)}{\partial Z} = -\frac{h_U L}{k} H(1) \tag{15}$$

$$\frac{\partial H(0)}{\partial Z} = \frac{h_B L}{k} H(0). \tag{16}$$

The dimensionless heat transfer coefficients $h_U L/k$ and $h_B L/k$ are the Biot numbers, Bi_U and Bi_B, respectively.

Equation (7) can be analyzed in many ways and any solution method is acceptable if the numerical accuracy is good and the computation time is reasonable. The power series expansion which Sparrow *et al.* [2] used is taken in this study.

Six homogeneous solutions $f^{(0)}(Z), f^{(1)}(Z), \ldots f^{(5)}(Z)$ are obtained from equation (7) and they are expanded in power series as,

$$F(Z) = \sum_{i=0}^{5} C_i f^{(i)}(Z)1 \tag{17}$$

$$f^{(i)}(Z) = \sum_{n=0}^{\infty} b_n^{(i)} Z^n. \tag{18}$$

This equation is introduced into equation (7) and the recurrence formula is obtained for the coefficient $b_n^{(i)}$. For $n \geqslant 6$,

$$b_n^{(i)} = \frac{3a^2}{n(n-1)} b_{n-2}^{(i)} - \frac{3a^4}{n(n-1)\ldots(n-3)} b_{n-4}^{(i)} + \frac{a^6}{n(n-1)\ldots(n-5)} b_{n-6}^{(i)} - \frac{a^2 Ra^0 [2Ns\, b_{n-7}^{(i)} + (1-Ns) b_{n-6}^{(i)}]}{n(n-1)\ldots(n-5)} \tag{19}$$

where $b_{n-7}^{(i)}$ is zero when $n < 7$. Linear independence among $f^{(i)}(Z)$ values is guaranteed by making $b_n^{(i)} = \delta_n^i$ (Kronecker delta) for $0 \leqslant n \leqslant 5$.

Normally, the coefficients C_i for $f^{(i)}(Z)$ are determined from the boundary conditions. Since the boundary conditions are all homogeneous, the coefficients C_i can be nontrivial only when the determinant of the coefficient matrix vanishes. Thus the objective is to find the eigenvalue Ra^0 which makes the determinant zero at a given value of a, and then to choose the least value of Ra^0 (the critical Rayleigh number) and the associated a.

Before proceeding further, it is worthwhile to check the meaning of Ns. Natural convection may occur only when the temperature gradient $\partial T_s/\partial z$ is negative globally or locally. This situation can never happen when $-1 \leqslant Ns \leqslant 0$ with $T_B < T_U$. When Ns is positive, $T_B > T_U$. The maximum temperature occurs at $Z = 0$ when $0 \leqslant Ns \leqslant 1$ (the maximum temperature and the location are notated as T_{max} and z_{max}). We define the modified Rayleigh number Ra as,

$$Ra = g\beta(T_{max} - T_U)(L - z_{max})^3/\alpha\nu. \tag{20}$$

When $0 \leqslant Ns \leqslant 1$, Ra is the same as the original Rayleigh number Ra^0. When $Ns \geqslant 1$ or $Ns < -1$, the maximum temperature occurs at $Z = \frac{1}{2} - (1/2Ns)$ and

$$T_{max} - T_U = \tfrac{1}{4}(T_B - T_U)(Ns+1)\left(1 + \frac{1}{Ns}\right) \tag{21}$$

thus making,

$$Ra = \tfrac{1}{32} Ra^0 Ns \left(1 + \frac{1}{Ns}\right)^5. \tag{22}$$

The factors influencing Ra are Ns, Bi_U and Bi_B.

The modified Rayleigh number was extensively obtained for all possible combinations of the following parameters; free or rigid upper surface: $Ns = \pm\infty, -10, -3, -2, -1.5, 0, 0.1, 0.3, 1, 3, 10$: $Bi_B = 0, 0.1, 0.3, 1, 3, 10, \infty$: $Bi_U = 0, 0.1, 0.3, 1, 3, 10, \infty$. When Ns is $\pm\infty$, the upper and the lower surfaces have a same temperature. When it is zero, there is no heat generation. When the Biot number is zero, the surface has a heat flux boundary condition, and when it is infinitely large, the surface temperature is fixed. For every combination of Ns, Bi_B and Bi_U, the wavenumber a was varied and the eigenvalue Ra was calculated using a modified linear interpolation method [3]. The wavenumber was varied in the external loop and the minimum Ra, i.e. Ra_c was found. Variation of a was made using a golden section search. When the variations of a and Ra were within 10^{-5} and 10^{-3}, respectively, it was assumed to have completely converged. Seventy terms were retained in computing $f^{(i)}(Z)$. Actual computation was made using a CRAY-2S computer. Computing time was very short and it was not a significant issue. Some computed results can be compared with those from other sources. Specifically, when $Ns = 0$ and the upper and the lower surfaces are rigid with fixed temperatures, computed Ra_c and a are 1707.762 and 3.1163, respectively. This result is in perfect agreement with the accurate results of Fujimura and Kelly (1707.762 and 3.116324, respectively) [4]. When $Ns \neq 0$, some limited results for rigid upper surface and $Bi_U = Bi_B = \infty$ are available for comparison [2]. Again the agreement is good within 2 digits below the decimal point.

When both of the upper and the lower Biot numbers were zero, calculation procedure frequently experienced difficulty in obtaining Ra_c. This happened as the critical wavenumber became zero, whose region will be given in the following section. If it happened, very small Bi_B (10^{-5}, 10^{-6}, 10^{-7} and 10^{-8}) with $Bi_U = 0$ was successively tried and the asymptotic behavior was examined by treating the critical Rayleigh number as an infinite sequence. This method was very accurate and the induced error of Ra_c was at most 2.4×10^{-4}. When Ns approaches -1 from the left, it corresponds to the limiting case of heat flow mainly to the bottom while the upper surface does not transmit any heat. The bottom is at lower temperature than the top and convection is difficult to occur. The originally defined critical Rayleigh number Ra_c^0 thus increases indefinitely. For this reason, numerical computation was performed up to $Ns = -1.5$.

RESULTS AND DISCUSSIONS

The computed results of Ra_c and a_c are compressed into Figs. 2 and 3 in 3D plots. Not all the computed results are presented in the figures for the following reasons. The critical Rayleigh number or the wavenumber for any Bi_B lies between the extreme cases of 0 and ∞, and the dependence on Bi_B is monotonic. Thus, these two extreme cases are shown.

Generally speaking, the modified critical Rayleigh number Ra_c is between 100 and 1700 for all the tested cases and it is greater for greater Bi_B. With a few minor exceptions, it is greater when Bi_U increases and when the upper surface is hydraulically rigid [see Figs. 2(a) and (b) and Figs. 3(a) and (b)]. Its variation with the Biot numbers is roughly within 50% of the maximum Ra_c.

When Ns is less than -1, Ra_c is found to be almost insensitive to the variation of Bi_U whether the upper surface is free or rigid. Though the originally defined critical Rayleigh number Ra_c^0 becomes very large as Ns approaches -1 from $-\infty$ with other parameters fixed, the modified critical Rayleigh number Ra_c does not change very much. It may be explained from the fact that the actual convection takes place mostly in the upper part where $\partial T_s/\partial z$ is negative. The thickness of the part of negative temperature gradient is decreased as Ns approaches -1, and Ra_c based on the reduced length is very insensitive to the change of Ns. When Ns is positive and less than unity, the maximum temperature

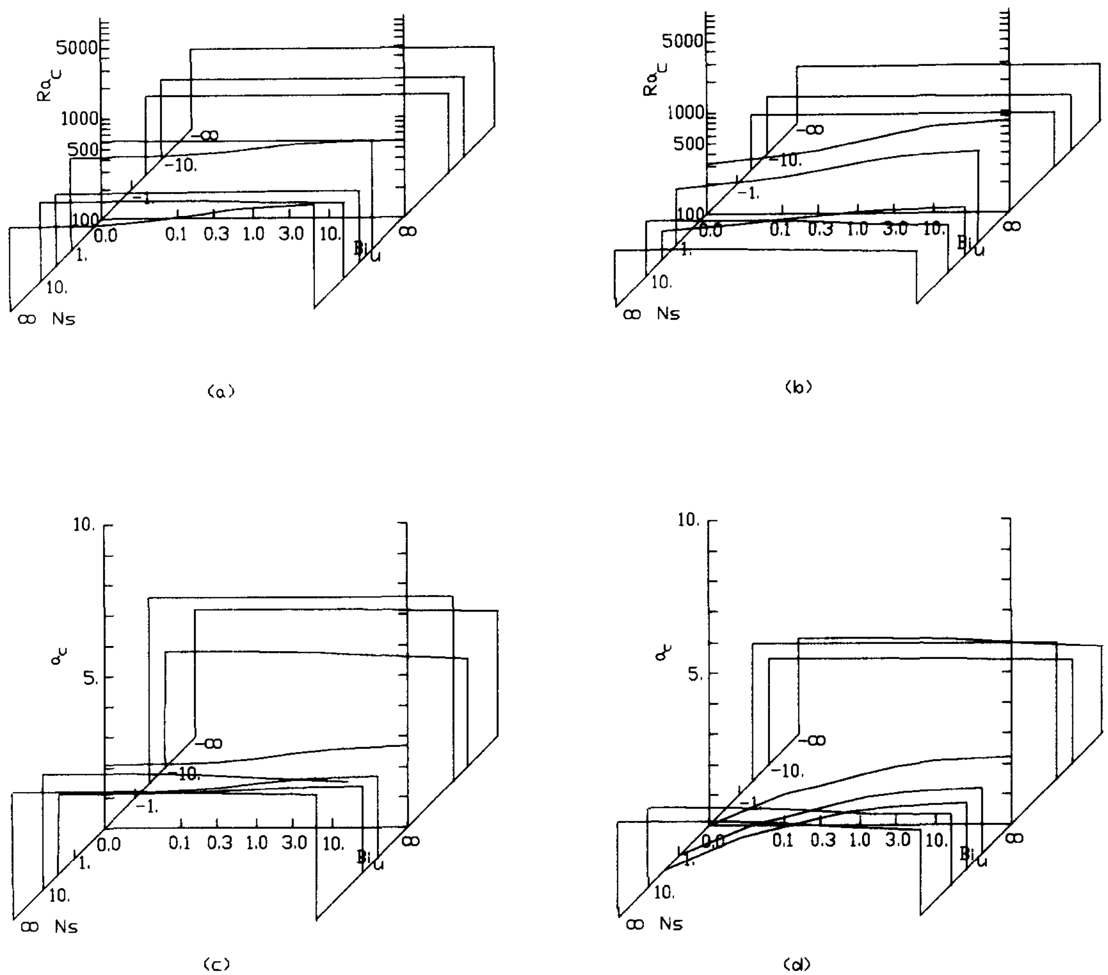

Fig. 2. The modified critical Rayleigh number (a) $Bi_B = \infty$ and (b) $Bi_{B=0}$ and the wavenumber (c) $Bi_B = \infty$ and (d) $Bi_B = 0$ for hydraulically free upper surface problem. (Note that Bi_U scale is not logarithmic in $0.1 \leqslant Bi_U \leqslant 10$, and it has no scale outside. Also, Ns has logarithmic scale in $1 \leqslant |Ns| \leqslant 10$ and it has no scale elsewhere. Identify the curves aligning the left ends vertically on the Ns axis.)

occurs at the bottom. The two critical Rayleigh numbers Ra_c and Ra_c^0 are the same. It increases with greater lower and/or upper Biot numbers. The effect of Ns is small in this region. As Ns grows farther beyond 1, it changes more or less monotonically and asymptotically. When $Ns = 10$ or greater, the upper surface Biot number has little effect on Ra_c, whether the upper surface is free or rigid. The hydraulic condition of the upper surface is insignificant in determining Ra_c when Ns is very large.

As shown in Figs. 2(c) and (d) and Figs. 3(c) and (d), the critical wavenumber a_c lies between 1 and 8 mostly (maximum is found to be about 12.5). It is greater when Bi_U is greater. No general trend is found regarding which hydraulic boundary condition at the upper surface gives greater a_c. A strange behavior has been found; a_c approaches zero when the upper and the lower Biot numbers are all zero and when $0 \leqslant Ns \leqslant 4.489$ with free upper surface, or when $0 \leqslant Ns \leqslant 4.701$ with rigid upper surface. This happens when the gradient of the eigenvalue Ra with respect to the wavenumber a is positive at $a = 0$. The asymptotic approach described in the previous section was applied to find Ra_c in this case. Note that $a_c = 0$ means very large convection cell.

When Ns approaches -1 from the negative direction, a_c becomes large. This means that the convection cells are small when they are induced by internal heat generation in a fluid layer with lower bottom temperature than the top temperature. This observation is also in accordance with that of Sparrow *et al.* [2].

CONCLUSIONS

A linear stability theory has been applied to a fluid layer with internal heat generation subject to upper and lower convection boundary conditions. Dimensionless heat generation number Ns has been used to transform the original critical Rayleigh number to a modified one (Ra_c). Regions of instability for Ns have been examined and Ra_c and a_c are computed for $\sigma = 0$ using power series expansion.

Generally, Ra_c is between 100 and 1700 for all tested cases and it is greater for greater Bi_B. It is roughly greater when Bi_U increases and when the upper surface is hydraulically rigid. The wavenumber a_c is between 1.0 and 12.5 for most cases. It is zero when the upper and the lower Biot numbers are zero and Ns lies between 0 and about 4.5.

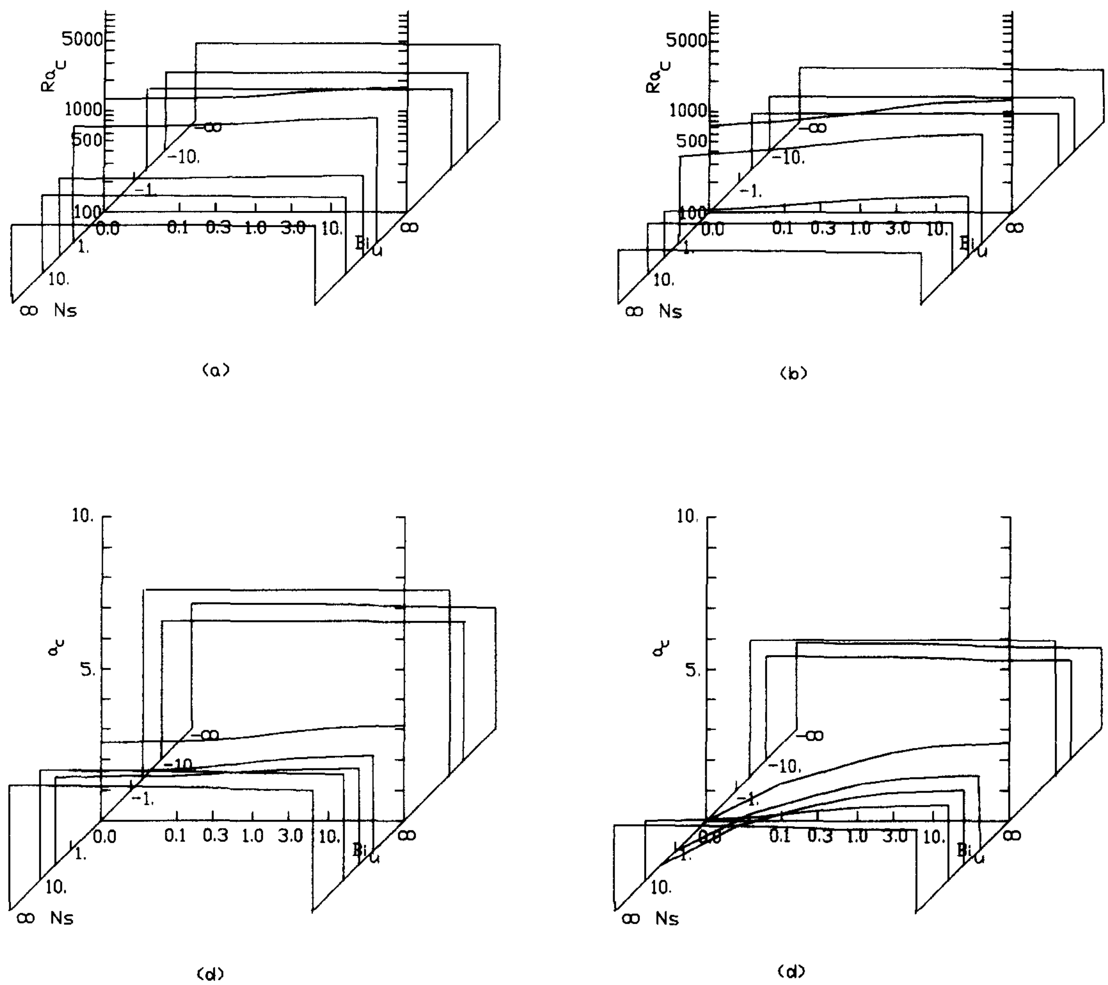

Fig. 3. The modified critical Rayleigh number (a) $Bi_B = \infty$ and (b) $Bi_{B=0}$ and the wavenumber (c) $Bi_B = \infty$ and (d) for $Bi_B = 0$ for hydraulically rigid upper surface problem. (Comments on the axes are the same as in Fig. 2.)

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